Computers and Mathematics with Applications 59 (2010) 1709–1717

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On analogues of exponential functions for antisymmetric fractional derivatives

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ARTICLE INFO

Keywords:
Fractional calculus
Riemann–Liouville derivative
Antisymmetric fractional derivative
Meijer G-functions series

ABSTRACT

An equation with the antisymmetric fractional derivative of order $\alpha \in (1, 2)$, containing the t^{β}-potential is solved using the Mellin transform method. The solutions are analogues of exponential functions of a new type. They are represented as Meijer G-function series in a finite time interval. In the classical limit $\alpha \longrightarrow 1^{+}$, the eigenfunction equation for a derivative of the first order and its solution – an exponential function, are recovered. Then an analogy between the derivation of Euler–Lagrange equations in fractional mechanics and in classical mechanics is discussed. The results are applied to a simple fractional Euler–Lagrange equation containing an antisymmetric fractional derivative and its general solution is obtained.

1. Introduction

Fractional operators appear as an important method in the description of various phenomena in physics, mechanics, chemistry, engineering, bioengineering and finance [1–7]. The application of fractional calculus to mathematical modelling yields fractional equations—both integral and differential ones. In the literature, ordinary and partial differential equations are studied, solved and applied (see monographs [8–10] and the references therein).

In recent years the Euler–Lagrange equations of fractional mechanics were obtained using the least action principle. The first results were derived by Riewe [11,12]. Then the Lagrangian and the Hamiltonian formulation of fractional mechanics were developed for models with a symmetric and an antisymmetric fractional derivative [13], subsequently for models with sequential fractional derivatives [14] and for models with constraints [15]. Interesting results were also discussed in papers [16–18] and lately also by Cresson in [19].

The characteristic feature of these equations of motion is the mixing of left- and right-sided Riemann–Liouville fractional derivatives. Therefore, these new classes of fractional differential equations become an interesting area of investigation. Certain equations of this type of fractional order $\alpha \in (0, 1)$ were studied in papers [20–23]. In the solution of fractional equations of variational type the composition rules of fractional calculus together with fixed point theorems where applied. Then the Mellin transform was proposed as a method of solving some equations including the composition of left- and right-sided derivatives [24]. The present paper is devoted to equations with an antisymmetric fractional derivative, which we define as the difference of the left- and right-sided derivatives. The aim of our investigations is the derivation of its stationary functions and analogues of exponential function. The developed method includes the application of the Riesz potential, composition rules and the Mellin transform.

The paper is organized as follows. In Sections 1.1 and 1.2 we review relevant definitions and formulas from fractional calculus. In Section 2 stationary functions for the antisymmetric derivative are obtained and their properties are studied. Section 3 contains the solution of a class of fractional equations with an antisymmetric derivative by means of the Mellin

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doi:10.1016/j.camwa.2009.08.013

transform. Then in Section 4 we discuss briefly fractional mechanics with the antisymmetric derivative and give an example of the application of earlier derived functions to solve certain variational equations of fractional mechanics.

1.1. Fractional operators

We recall some definitions of fractional operators and briefly discuss their properties. The fractional integrals of order $\alpha \in R_+$ are defined as follows [10,25]:

$$I^{\alpha}_{0+}f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(u)\mathrm{d}u}{(t-u)^{1-\alpha}} \quad t > 0 \tag{1}$$

$$I^{\alpha}_{b-}f(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{f(u)\mathrm{d}u}{(u-t)^{1-\alpha}} \quad t < b \tag{2}$$

$$I^{\alpha}_{-}f(t) := \frac{1}{\Gamma(\alpha)}\int_t^{\infty} \frac{f(u)\mathrm{d}u}{(u-t)^{1-\alpha}} \quad t < \infty. \tag{3}$$

The first two integrals I^{α}_{0+} and I^{α}_{b-} are known as the left-sided and respectively the right-sided Riemann–Liouville fractional integral and the last one is called the right-sided Liouville integral.

The Riemann–Liouville derivatives are constructed using the above fractional integrals. For real order $\alpha \in (n-1, n)$ they look as follows (we have denoted the classical derivative as $D := \frac{\mathrm{d}}{\mathrm{d}t}$) [10,25]:

$$D^{\alpha}_{0+}f(t) := D^n I^{n-\alpha}_{0+}f(t) \tag{4}$$

$$D^{\alpha}_{b-}f(t) := (-D)^n I^{n-\alpha}_{b-}f(t). \tag{5}$$

In limit $\alpha \longrightarrow n^+$ for $\alpha \in (n, n+1)$ we recover the classical derivatives of the nth order.

The Riemann–Liouville derivatives obey the integration by parts formula which mixes the left- and right-sided derivatives [25]. Thus, the Euler–Lagrange equations in fractional mechanics obtained using this formula always contain both types of derivatives.

Another useful property of fractional operators are their composition rules, which we shall apply in the derivation of an analogue of an exponential function. For any function $f \in L_1(R_+)$ we have the following relations valid almost everywhere in interval $[0, b]$ [10]:

$$D^{\alpha}_{0+}I^{\alpha}_{0+}f = f \qquad D^{\alpha}_{-}I^{\alpha}_{-}f = f \tag{6}$$

$$D^{\beta}_{0+}I^{\alpha}_{0+}f = I^{\alpha-\beta}_{0+}f \qquad D^{\beta}_{-}I^{\alpha}_{-}f = I^{\alpha-\beta}f \tag{7}$$

when $\alpha > \beta > 0$. Let us observe that for function $f \in C[0, b]$ the above composition rules are fulfilled at any point $t \in [0, b]$. In turn, when $t \in C_{\gamma}[0, b]$ they are fulfilled for $t \in (0, b]$.

In the following calculations we shall also use the notion of a modified Riesz potential given by the integral on the real half axis with a power function kernel:

$$H^{\alpha}_0 f(t) :== \frac{1}{2\Gamma(\alpha)\sin(\pi\alpha/2)}\int_0^{\infty} \frac{f(u)\cdot \mathrm{sign}(t-u)}{|\, t-u\,|^{1-\alpha}}\mathrm{d}u. \tag{8}$$

Using the Mellin transform we can prove the following important property of this potential (compare also Theorem 12.4 from the monograph by Samko et al [25]). This theorem states that for $\alpha \in (0, 1)$ and $f \in L_p(R_+)$ with $p \in [1, 1/\alpha)$ we can rewrite the modified Riesz potential as follows:

$$H^{\alpha}_0 f = -DI^{(1+\alpha)/2}_{-}I^{(1+\alpha)/2}_{0+}f. \tag{9}$$

The above formula will be the main instrument in the derivation of stationary functions and then in the solution of fractional equations with an antisymmetric derivative and variable potential.

1.2. Mellin transform and its properties

We propose to apply one of the integral transforms as the derivation method of an analogue of the exponential function for an antisymmetric fractional derivative. We shall use the Mellin transform which looks as follows for sufficiently good functions [26]:

$$\mathcal{M}[f](s) := \int_0^{\infty} t^{s-1}f(t)\mathrm{d}t. \tag{10}$$

Similarly to the Laplace transform, the Mellin transform also has its convolution defined by the formula:

$$f * g(t) := \int_0^\infty f(u) g\left(\frac{t}{u}\right) \frac{\mathrm{d}u}{u}. \tag{11}$$

When the Mellin transform acts on the Mellin convolution of two functions, the result is the multiplication of the corresponding transforms of both functions:

$$\mathcal{M}[f * g](s) = \mathcal{M}[f](s) \cdot \mathcal{M}[g](s). \tag{12}$$

We shall also apply the following shifting property of the Mellin transform:

$$\mathcal{M}[t^\beta f(t)](s) = \mathcal{M}[f(t)](s + \beta). \tag{13}$$

Finally, we quote a Lemma describing the Mellin transform for fractional integrals and derivatives from the monograph by Kilbas et al. [10].

Lemma 1.1. (1) *If* $Re(s - \alpha) < 1$ *and the following limits vanish*

$$\lim_{t \longrightarrow 0^+} [t^{s-k-1} I_{0+}^{n-\alpha} f(t)] = 0, \qquad \lim_{t \longrightarrow \infty} [t^{s-k-1} I_{0+}^{n-\alpha} f(t)] = 0$$

for $k = 0, \dots, n-1$ *then the following formula for the Mellin transform of the left-sided fractional derivative is valid:*

$$\mathcal{M}[D_{0+}^\alpha f](s) = \frac{\Gamma(1 - (s - \alpha))}{\Gamma(1 - s)} \mathcal{M}[f](s - \alpha). \tag{14}$$

(2) *If* $Re(s) > 0$ *and the following limits vanish*

$$\lim_{t \longrightarrow 0^+} [t^{s-k-1} I_{-}^{n-\alpha} f(t)] = 0, \qquad \lim_{t \longrightarrow \infty} [t^{s-k-1} I_{-}^{n-\alpha} f(t)] = 0$$

for $k = 0, \dots, n-1$ *then the following formula for the Mellin transform of the right-sided fractional derivative is valid:*

$$\mathcal{M}[D_{-}^\alpha f](s) = \frac{\Gamma(s)}{\Gamma(s - \alpha)} \mathcal{M}[f](s - \alpha). \tag{15}$$

(3) *Let* $s \in C$ *and* $\int_0^\infty \mid t^{s+\alpha-1} f(t) \mid \mathrm{d}t < \infty$*. The following formula holds for* $Re(s) < 1 - \alpha$*:*

$$\mathcal{M}[I_{0+}^\alpha f](s) = \frac{\Gamma(1 - \alpha - s)}{\Gamma(1 - s)} \mathcal{M}[f](s + \alpha). \tag{16}$$

(4) *Let* $s \in C$ *and* $\int_0^\infty \mid t^{s+\alpha-1} f(t) \mid \mathrm{d}t < \infty$*. The following formula holds for* $Re(s) > 0$*:*

$$\mathcal{M}[I_{-}^\alpha f](s) = \frac{\Gamma(s)}{\Gamma(s + \alpha)} \mathcal{M}[f](s + \alpha). \tag{17}$$

2. Stationary functions for antisymmetric fractional derivative of order $\alpha \in (1, 2)$

We shall consider the antisymmetric fractional derivative in finite time interval $[0, b]$ defined as the following linear combination of left- and right-sided Riemann–Liouville derivatives:

$$\mathcal{D}^\alpha := \frac{1}{2}[D_{0+}^\alpha - D_{b-}^\alpha]. \tag{18}$$

Let us note that in the limit when $\alpha \longrightarrow 1^+$ we recover the classical first order derivative. We shall solve a class of equations containing the antisymmetric derivative and the t^β- potential. The first step, however, is the explicit calculation of the stationary functions. These functions fulfill the following equation in finite time interval $[0, b]$:

$$\mathcal{D}^\alpha f_\alpha^{st}(t) = 0 \quad t \in [0, b] \tag{19}$$

and they extend the notion of polynomial functions which are stationary functions for integer order derivatives. Applying definitions (4) and (5) we transform the above equation to its integral form:

$$\frac{1}{2}[I_{0+}^{2-\alpha} - I_{b-}^{2-\alpha}] f_\alpha^{st}(t) = P_1(t) \Delta H(t), \tag{20}$$

where function P_1 is an arbitrary polynomial of the first degree and ΔH is the difference of Heaviside's functions $\Delta H(t) := H(t) - H(t - b)$. Let us denote constant $S_\alpha := \sin(\pi\alpha/2)$ and we observe that the above fractional integral operator is proportional to the modified Riesz potential.

When we extend the stationary function to $t \in R_+$ assuming $f_\alpha^{st}(t) \equiv f_\alpha^{st}(t)\Delta H(t)$, then Eq. (20) can be reformulated as follows:

$$S_\alpha H_0^{2-\alpha} f_\alpha^{st}(t) = P_1(t)\Delta H(t). \tag{21}$$

The next step is the application of property (9) for the Riesz potential, which gives the equation:

$$S_\alpha I_-^{\frac{3-\alpha}{2}} I_{0+}^{\frac{3-\alpha}{2}} f_\alpha^{st}(t) = P_2(t)\Delta H(t) \tag{22}$$

with P_2 being an arbitrary polynomial of the second degree.

The above equation can be solved using composition rules (6):

$$f_\alpha^{st}(t) = \frac{1}{S_\alpha} D_{0+}^{\frac{3-\alpha}{2}} D_-^{\frac{3-\alpha}{2}} P_2(t)\Delta H(t). \tag{23}$$

Then we take polynomial function

$$P_2(t) = \sum_{k=0}^{2} A_k (b-t)^k$$

with arbitrary constant coefficients A_k and by means of straightforward calculation we obtain the following stationary function for antisymmetric fractional derivative $\mathcal{D}^\alpha$ in finite time interval $[0, b]$:

$$f_\alpha^{st}(t) = \frac{1}{S_\alpha}(bt)^{\frac{\alpha-3}{2}} \sum_{k=0}^{2} A_k \Gamma(k+1) b^k {}_1\Psi_2 \left[\begin{matrix} (1,1) \\ \left(\frac{\alpha-1}{2}, 1\right) \quad \left(\frac{\alpha-1}{2}+k, -1\right) \end{matrix} \mid -\frac{t}{b} \right]. \tag{24}$$

The above stationary function contains power function $t^{(\alpha-3)/2}$, singular at $t = 0$ and an arbitrary linear combination of the Fox–Wright functions (see for example defining formula (1.11.14) in the monograph by Kilbas et al. [10]). In a general case it includes three arbitrary real or complex coefficients A_k.

2.1. Properties of stationary functions f_α^{st}

Let us point out that for arbitrary coefficients the above stationary function (24) can be singular at the ends of time interval $[0, b]$. However, we can ensure the continuity of f_α^{st} at the beginning of time interval $t = 0$. It requires adding the following condition for coefficients:

$$\sum_{k=0}^{2} A_k b^k \frac{\Gamma(k+1)}{\Gamma(k+\frac{\alpha-1}{2})} = 0. \tag{25}$$

Let us now check the behaviour of our stationary function at end $t = b$. It is determined by properties of the Fox–Wright functions given by the parameters:

$$\Delta = -1 \qquad \delta = 1 \qquad \mu = \alpha + k - \frac{5}{2}. \tag{26}$$

The Fox–Wright function is continuous at $t = b$ (compare Theorem 1.5 from the monograph by Kilbas et al. [10]) provided the condition

$$Re(\mu) > \frac{1}{2} \Longrightarrow k = 2 \tag{27}$$

is fulfilled.

Hence, assuming $A_0 = A_1 = 0$ we obtain a solution continuous at $t = b$.

Concluding, we can consider the stationary functions of general form (24) dependent on three arbitrary constants. If we restrict the set of stationary functions to functions continuous at end $t = 0$, we have two arbitrary constants. In what follows, we shall use the $k = 2$ component, which means the stationary function will be in the form of

$$f_\alpha^{st}(t) = \frac{2b^2 A_2}{S_\alpha}(bt)^{\frac{\alpha-3}{2}} {}_1\Psi_2 \left[\begin{matrix} (1,1) \\ \left(\frac{\alpha-1}{2}, 1\right) \quad \left(\frac{\alpha-1}{2}+2, -1\right) \end{matrix} \mid -\frac{t}{b} \right] \tag{28}$$

and it belongs to the $C_{(3-\alpha)/2}[0, b]$ space (compare formula (1.1.21) in [10]).

Further analyzing the properties of stationary function (24) we arrive at a proper classical limit as the following statements are valid:

$$\alpha \longrightarrow 1^+ \Longrightarrow \mathcal{D}^\alpha \longrightarrow D \equiv \frac{\mathrm{d}}{\mathrm{d}t} \tag{29}$$

$$\alpha \longrightarrow 1^+ \Longrightarrow f_\alpha^{st} \longrightarrow 2A_2 \tag{30}$$

and we recover the classical derivative of the first order with its stationary function - an arbitrary constant.

3. Mellin transform in solution of fractional equations with antisymmetric derivative and t^β- potential

We shall now derive function $\mathcal{E}_{\alpha,\lambda}$ solving the following fractional differential equation with an antisymmetric fractional derivative:

$$[\mathcal{D}^\alpha - \lambda t^\beta]\mathcal{E}_{\alpha,\lambda}(t) = 0, \tag{31}$$

where $\lambda \in C$ is an arbitrary complex number, $\alpha \in (1,2)$, $\beta \in R$ and Eq. (31) is fulfilled in finite time interval $[0, b]$.

The integral version of the above equation can be obtained in a procedure similar to that presented in the previous section and it looks as follows for $t \in [0, b]$:

$$\left[S_\alpha I_{0+}^{\frac{3-\alpha}{2}} - \lambda I_-^{\frac{3+\alpha}{2}} t^\beta\right]\mathcal{E}_{\alpha,\lambda}(t) = D_-^{\frac{3-\alpha}{2}} P_2(t)\Delta H(t) \tag{32}$$

and we assume $\mathcal{E}_{\alpha,\lambda}(t) \equiv \mathcal{E}_{\alpha,\lambda}(t)\Delta H(t)$.

In the next step we apply composition rule (6) and we rewrite the above equation as follows:

$$\left[1 - \Lambda D_{0+}^{\frac{3-\alpha}{2}} I_-^{\frac{3+\alpha}{2}} t^\beta\right]\mathcal{E}_{\alpha,\lambda}(t) = f_\alpha^{st}(t)\Delta H(t), \tag{33}$$

where the stationary function is given in formula (23) and (28) and $\Lambda = \lambda/S_\alpha$.

After the application of the Mellin transform, the transformed version of Eq. (33) is fulfilled on the strip of the complex plane given by condition $Re(s) \in \left(\frac{3-\alpha}{2}, 1 + \frac{3-\alpha}{2}\right)$ and it has the following form:

$$\left[1 - \Lambda g(s) T_{\alpha+\beta}\right]\mathcal{M}[\mathcal{E}_{\alpha,\lambda}](s) = \mathcal{M}[f_\alpha^{st}(t)\Delta H(t)](s), \tag{34}$$

where we have used the notation:

$$g(s) = \frac{\Gamma(s - \frac{3-\alpha}{2})\Gamma(1 - (s - \frac{3-\alpha}{2}))}{\Gamma(1-s)\Gamma(s+\alpha)}. \tag{35}$$

Operator T appearing in Eq. (34) is the translation operator acting as follows on the functions of complex variable s- $T_{\alpha+\beta} v(s) := v(s + \alpha + \beta)$.

Hence, the above Eq. (34) is a difference equation - one of functional iterative equations described in the monograph [27]. It is solved explicitly by the following series:

$$\mathcal{M}[\mathcal{E}_{\alpha,\lambda}](s) = \sum_{m=0}^{\infty} \Lambda^m [g(s) T_{\alpha+\beta}]^m \mathcal{M}[f_\alpha^{st}\Delta H](s) \tag{36}$$

which is absolutely convergent for each complex number s taken from strip $Re(s) \in \left(\frac{3-\alpha}{2}, 1 + \frac{3-\alpha}{2}\right)$. Let us note that for any given fixed real part $Re(s) = \gamma$ the convergence is also uniform with respect to the imaginary part of the complex numbers taken from this straight line.

Now we calculate the inverse Mellin transform. We start with the last terms on the right-hand side of equality (36). Shifting property (13) indicates that their inverse Mellin transform is given by the formula:

$$\mathcal{M}^{-1}\left[T_{m(\alpha+\beta)}\mathcal{M}[f_\alpha^{st}\Delta H]\right](t) = t^{m(\alpha+\beta)} f_\alpha^{st}(t)\Delta H(t), \tag{37}$$

where the stationary function is described by (28).

Each of the middle terms in series (36) involves the product of the g-functions. We note that they are analogous in form to the Mellin transform of Meijer G-functions:

$$\prod_{l=0}^{m-1} g(s + l\alpha) = \mathcal{G}_{2m,2m}^{m,m}\left[\begin{matrix}\mathbf{A}_m \\ \mathbf{B}_m\end{matrix} \mid s\right], \tag{38}$$

where we have denoted vectors $\mathbf{A}_m$ and $\mathbf{B}_m$ as follows:

$$\mathbf{A}_m = \left[-\frac{3-\alpha}{2}\mathbf{e}_m + (\alpha+\beta)\mathbf{j}_m, \alpha\mathbf{e}_m + (\alpha+\beta)\mathbf{j}_m\right] \in R^{2m} \tag{39}$$

$$\mathbf{B}_m = \left[-\frac{3-\alpha}{2}\mathbf{e}_m + (\alpha+\beta)\mathbf{j}_m, (\alpha+\beta)\mathbf{j}_m\right] \in R^{2m} \tag{40}$$

with vectors $\mathbf{e}_m$ and $\mathbf{j}_m$ given as

$$\mathbf{e}_m = [1, \ldots, 1] \in R^m, \qquad \mathbf{j}_m = [0, 1, \ldots, m-1].$$

We check the parameters of kernels $\mathcal{G}$ to prove the existence of vertical contour $\mathcal{L}_{i\gamma\infty}$ necessary in the inversion procedure:

$$\Delta = 0 \qquad \delta = 1 \qquad a^* = 0 \qquad \mu = -m\alpha \quad m \in N. \tag{41}$$

Further, we also apply representation Theorem 3.3 from the monograph by Kilbas and Saigo [28] which gives the condition for parameters κ, ϑ separating the poles in the numerators of kernels $\mathcal{G}$. For $\alpha + \beta > 0$ we obtain

$$\kappa = \epsilon_a \quad \vartheta = 1 + \epsilon_a - (m-1)(\alpha + \beta) \tag{42}$$

and consequently for $\alpha + \beta < 0$

$$\kappa = \epsilon_a - (m-1)(\alpha + \beta) \quad \vartheta = 1 + \epsilon_a, \tag{43}$$

where we denoted $\epsilon_a := (3 - \alpha)/2$.

According to the mentioned theorem on representation of Fox and Meijer functions, vertical contour $\mathcal{L}_{i\gamma\infty}$ exists when the following set of conditions is fulfilled:

$$\kappa < \vartheta \tag{44}$$

$$\Delta\gamma + Re(\mu) < -1 \Longrightarrow \alpha > 1. \tag{45}$$

As we are studying equations of order $\alpha \in (1, 2)$ we clearly see that the second condition is fulfilled for any $m \in N$.

On the other hand the condition separating the poles requires further restrictions on order α and power exponent β. It appears that the respective assumptions should be chosen as follows

$$\alpha + \beta = \frac{\epsilon_a}{J} \quad \text{for } \alpha + \beta > 0 \tag{46}$$

$$\alpha + \beta = -\frac{\epsilon_a}{J} \quad \text{for } \alpha + \beta < 0, \tag{47}$$

where $J \in N$ is an arbitrary positive integer number.

In what follows we shall discuss case $\alpha + \beta > 0$ and $J = 1$. Such an assumption implies that the reduction formula (2.1.1) from the monograph [28] can be applied to kernels $\mathcal{G}^{m,m}_{2m,2m}$, namely the following equalities hold for any $m \in N$

$$\prod_{l=0}^{m-1} g(s + l(\alpha + \beta)) = \mathcal{G}^{m,1}_{m+1,m+1}\left[\begin{matrix}\mathbf{A}^a_m \\ \mathbf{B}^a_m\end{matrix} \mid s\right] \tag{48}$$

with vectors $\mathbf{A}^a_m, \mathbf{B}^a_m \in R^{m+1}$ given by formulas

$$\mathbf{A}^a_m = \left[-\epsilon_a, \alpha\mathbf{e}_m + \epsilon_a\mathbf{j}_m\right] \tag{49}$$

$$\mathbf{B}^a_m = \left[-\epsilon_a\mathbf{e}_m + \epsilon_a\mathbf{j}_m, (m-1)\epsilon_a\right]. \tag{50}$$

We conclude that for $\alpha \in (1, 2)$ and β obeying condition $\alpha + \beta = \epsilon_a$ we obtain the solution of Eq. (31) in the form of series of Meijer G-functions

$$\mathcal{E}_{\alpha,\lambda}(t) = f^{st}_\alpha(t)\Delta H(t) + \sum_{m=1}^{\infty} \Lambda^m G^{m,1}_{m+1,m+1}\left[\begin{matrix}\mathbf{A}^a_m \\ \mathbf{B}^a_m\end{matrix} \mid t\right] * t^{m\epsilon_a} f^{st}_\alpha(t)\Delta H(t), \tag{51}$$

where stationary function f^{st}_α is given by formula (28).

All the above considerations yield the following Proposition valid.

Proposition 3.1. *Let $\alpha \in (1, 2)$ and $\alpha + \beta = \epsilon_a$. Then an equation with antisymmetric fractional derivative $\mathcal{D}^\alpha$:*

$$(\mathcal{D}^\alpha - \lambda t^\beta)\mathcal{E}_{\alpha,\lambda}(t) = 0 \quad t \in [0, b]$$

has for any $\lambda \in C$ a general solution in the form of

$$\mathcal{E}_{\alpha,\lambda}(t) = f^{st}_\alpha(t)\Delta H(t) + \sum_{m=1}^{\infty} \Lambda^m G^{m,1}_{m+1,m+1}\left[\begin{matrix}\mathbf{A}^a_m \\ \mathbf{B}^a_m\end{matrix} \mid t\right] * t^{m\epsilon_a} f^{st}_\alpha(t)\Delta H(t), \tag{52}$$

where $\Lambda := \lambda/\sin(\pi\alpha/2)$, $\epsilon_a = (3 - \alpha)/2$ and stationary functions f^{st}_α are given by (28). Vectors $\mathbf{A}^a_m$ and $\mathbf{B}^a_m$ are given by formulas (49) and (50).

Solution (52) belongs to the $C_{\epsilon_a}[0, b]$ space which means it is continuous in any closed subinterval of $(0, b]$, provided $\mid \Lambda \mid < \mathrm{e}^{-\alpha} b^{-\epsilon_a}$.

Let us note that the above Proposition describes general solutions of Eq. (31) for case $\alpha + \beta = \epsilon_a$. Analogous statements are valid for order α and power exponent β fulfilling the more general condition $\alpha + \beta = \epsilon_a/J, J \in N$. The boundary conditions and resulting particular solutions are still under investigation.

3.1. Classical limit $\alpha \longrightarrow 1^+$

We derived an explicit analytical expression for the analogues of exponential functions solving Eq. (31) with an antisymmetric fractional derivative. They are given as the series of Meijer G-functions in the convolution with the respective stationary functions. Let us now discuss the classical limit when $\alpha \longrightarrow 1^+$. Then $S_1 = 1$ and the product of the g- functions looks as follows:

$$\prod_{l=0}^{m-1} g(s+l\alpha) = \frac{(-1)^m \Gamma(s)}{\Gamma(s+m)}. \tag{53}$$

The inverse Mellin transform gives the following formula for the $\mathcal{E}_{1,\lambda}$ function

$$\mathcal{E}_{1,\lambda}(t) = D(-D)P_2(t)\Delta H(t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{\Gamma(m)} \mathcal{M}^{-1}\left[\frac{\Gamma(s)\Gamma(m)}{\Gamma(s+m)}\right] * D(-D)P_2(t)\Delta H(t). \tag{54}$$

After some calculations the above function coincides with the known formula for the exponential function:

$$\mathcal{E}_{1,\lambda}(t) = C_1 \exp[\lambda(t-b)] \tag{55}$$

with C_1 being an arbitrary constant coefficient.

4. Euler–Lagrange equations in fractional mechanics with antisymmetric derivative $\mathcal{D}^\alpha$

Let us now compare the integration by parts formulas in classical and fractional calculi and the resulting Euler–Lagrange equations of simple mechanical models. For the first order derivative we have the formula:

$$\int_0^b f(t) \cdot Dg(t)\mathrm{d}t = \int_0^b g(t) \cdot (-D)f(t)\mathrm{d}t + f(t)g(t) \,|_{t=0}^{t=b} \tag{56}$$

valid for any pair of functions f, g differentiable in interval $[0, b]$.

In classical mechanics the above integration by parts formula yields for an action:

$$S = \int_0^b L(t, f, Df, \ldots, D^N f)\ \mathrm{d}t$$

the Euler–Lagrange equation of the following form:

$$\frac{\partial L}{\partial f} + \sum_{k=1}^{N} (-D)^k \frac{\partial L}{\partial D^k f} = 0 \tag{57}$$

valid for $t \in [0, b]$.

Let us now recall briefly the procedure of deriving Euler–Lagrange equations for a model in which the action will depend on the $\mathcal{D}^\alpha$ derivative of coordinate function f and on its sequential powers $(\mathcal{D}^\alpha)^k f$. Such models are studied in detail in the paper by Klimek [13], where the equations with the symmetric and antisymmetric derivatives of arbitrary order $\alpha \in (n-1, n)$ are considered.

For antisymmetric derivative $\mathcal{D}^\alpha$ we assume from the monograph by Samko et al. [25] the conditions: $\alpha \in (n-1, n)$ and $\frac{1}{p} + \frac{1}{q} < 1 + \alpha$, functions $f, g \in I^\alpha_{b-}[L_p] \cap I^\alpha_{0+}[L_q]$. Then the following fractional integration by parts formula is fulfilled:

$$\int_0^b f(t) \cdot \mathcal{D}^\alpha g(t)\mathrm{d}t = \int_0^b g(t) \cdot (-\mathcal{D}^\alpha)f(t)\mathrm{d}t. \tag{58}$$

We notice that formula (58) is analogous to (56). Hence, applying the least action principle to an action:

$$S = \int_0^b L(t, f, \mathcal{D}^\alpha f, \ldots, (\mathcal{D}^\alpha)^N f)\ \mathrm{d}t,$$

we arrive at the fractional Euler–Lagrange equation:

$$\frac{\partial L}{\partial f} + \sum_{k=1}^{N} (-\mathcal{D}^\alpha)^k \frac{\partial L}{\partial (\mathcal{D}^\alpha)^k f} = 0 \tag{59}$$

valid for $t \in [0, b]$ and analogous to equations from classical mechanics (57).

We know from the theory of ordinary linear differential equations, that the exponential functions of real or complex arguments can be applied in solving such equations, at least in the case of equations with constant coefficients.

Thus, the derived $\mathcal{E}_{\alpha,\lambda}$ functions will be essential in the solution of similar equations in which we shall replace the first order derivative with antisymmetric fractional derivative $\mathcal{D}^\alpha$. In the next section we shall solve an example of this type.

4.1. On simple application of $\mathcal{E}_{\alpha,\lambda}$ functions to variational equations of fractional mechanics

As we have mentioned the obtained $\mathcal{E}_{\alpha,\lambda}$- functions can be applied to solve a class of linear fractional differential equations dependent on the antisymmetric fractional derivative. Let us investigate here only one example. We consider an action with $D_0^{\alpha,\alpha} \equiv \mathcal{D}^\alpha$ derivative defined in papers by Klimek and Cresson [13,19]:

$$S = \int_0^b \left[\frac{t^{-\beta}}{2} (D_0^{\alpha,\alpha} f)^2 - \frac{\omega^2 t^\beta}{2} f^2 \right] \mathrm{d}t. \tag{60}$$

Using general formula (59) for the Euler–Lagrange equation we arrive at the following simple variational equation

$$[(t^{-\beta} D_0^{\alpha,\alpha})^2 + \omega^2] f(t) = [(t^{-\beta} \mathcal{D}^\alpha)^2 + \omega^2] f(t) = 0 \tag{61}$$

fulfilled in finite time interval $[0, b]$.

Let us observe that when $\alpha \longrightarrow 1^+$, the above equation becomes a harmonic oscillator equation

$$[D^2 + \omega^2] f(t) = 0 \tag{62}$$

as parameter $\beta = -\alpha + \epsilon_a = -\frac{3}{2}\alpha + \frac{3}{2} \longrightarrow 0$.

We apply the formulas for the analogues of the exponential functions from Proposition 3.1 and obtain the general solution for variational problem (61) in finite time interval $[0, b]$:

$$f(t) = C_1 \mathcal{E}_{\alpha,\mathrm{i}\omega}(t) + C_2 \mathcal{E}_{\alpha,-\mathrm{i}\omega}(t), \tag{63}$$

where C_1 and C_2 are arbitrary constants and condition $\alpha + \beta = \epsilon_a = (3 - \alpha)/2$ is fulfilled.

5. Final remarks

We derived the stationary functions and analogues of exponential functions for the antisymmetric fractional derivative of order $\alpha \in (1, 2)$. The proposed method includes the composition properties of fractional operators and the Mellin transform. A similar method was applied to other classes of fractional equations with mixed derivatives [24]. In all the studied cases the obtained solutions are given as Meijer G-functions series in the Mellin convolution with the respective stationary functions.

Although the presented results describe the case of real order $\alpha \in (1, 2)$, it is clear that they can be extended to an arbitrary real or more general - complex value of α.

The derived $\mathcal{E}_{\alpha,\lambda}$- functions were applied in solving the Euler–Lagrange equation for a certain simple model of fractional mechanics. This application will be extended in the subsequent paper to linear equations of the form

$$L(t^{-\beta} \mathcal{D}^\alpha) f(t) = 0 \quad t \in [0, b], \tag{64}$$

where operator L is given as follows:

$$L(t^{-\beta} \mathcal{D}^\alpha) := (t^{-\beta} \mathcal{D}^\alpha)^N + \sum_{k=0}^{N-1} a_k (t^{-\beta} \mathcal{D}^\alpha)^k \tag{65}$$

with arbitrary constant coefficients a_k $k = 1, \dots, N - 1$.

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